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**On the Nature of Impulsive Differential Equations and the Existence of its Solutions. The Emerging facts.**

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ABSTRACT

Examples of delayed impulsive differential equations show that the delay converts a differential equation with smooth right side to one with measurable right side. A traditional impulsive differential equation describing the impacts of impulses can be accompanied by another one that describes the impulse process. In order to support handling impulsive differential equations with delay, we formulated and proved existence theorems for impulsive differential equations with measurable right sides following Caratheodory's techniques. The new setup had impact on the formulation of initial value problems (IVP), the continuation of solutions and the structure of the system of trajectories. (a) We have two impulsive differential equations to solve with one IVP ($\varphi(\sigma_0) = \xi_0$) which selects one of the impulsive differential equations by the position of $\sigma_0$ in $[a, b_\nu]$. Solving the selected IVP fully determines the solution on the other scale with a possible delay. (b) The solutions can be continued at each point of $(\alpha,\ \beta) \times \Omega_0 =: \Omega$ by the conditions in the existence theorem. However, the jump at a discontinuity point may land outside $\Omega$. Thus, there is no continuation of the solution from such points of $\Omega$. If we restrict the jumps into $\Omega$, then all trajectories reaching a discontinuity point will be continued. If range of jumps $\neq \Omega$ then trajectories from outside the range of jumps cannot be continued backwards. (c) These changes alter the flow of solutions into a directed tree. This tree however is an in-tree which offers a modelling tool to study interactions of generations.

## INTRODUCTION

In this paper we consider some problems arising from the formulation of impulsive systems with delay in particular. As a response to problems arising from delay in impulsive systems, theorems about the existence of their solutions are formulated and proved. The theory of impulsive systems as an independent area of mathematical analysis is relatively new. The field started as a response to the observation that some physical and biological processes may also be influenced by short-time perturbations (impulses). The innovation of the theory of impulsive systems is manifested in the fact that the time-development of the state of such a system forms a mapping of bounded variation instead of continuously differentiable solution of a differential equation (Bainov and Simeonov, 1995, Ballinger, 1999, Lakshmikantham et al., 1989, Oyelami, 1999, Samoilenko and Perestyuk, 1995).

This paper gives a summary of some follow ups of results presented in two Ph.D. theses written and defended by the two co-authors.

**Systems Described by Impulsive Differential equations**

The Bainovian model is a dynamical system controlled by an absolute continuous term and a term of impulse effect. In formal terms: Let the process evolve in a period of time $T := (\alpha,\ \beta) \subset R$. Let $\Omega_0 \subset R^n$ be an open set and $\Omega := T \times \Omega_0$. Let $f : \Omega \to R^n$ be an at least continuous mapping which in addition may fulfil local Lipschitz condition in its variable $x \in R^n$ for each fixed t, $\forall\ (t, x) \in \Omega$. Let $H \subset Z$ be an infinite subset of $Z$ ($H = \mathbb{N}$ or $H = Z$ will be used). Then let the real time sequence $S_H = \{t_k\}_{k \in H} \subset T$ be increasing without finite accumulation points and $t_k \to \pm\infty,\ k \to \pm\infty$. Let $g : S_H \times R^n \to R^n$ be continuous, maybe Lipschitz function in its variable x, $\forall\ (t_k, x) \in \Omega$. Then the controlling impulsive differential equation is given by:

$$\begin{cases} x'(t) = f(t, x(t)), & \forall t \in T \setminus S_H \\ \Delta x(t_k) = g(t_k, x(t_k)), & \forall t_k \in S_H, \\ \Delta x(t_k) := x(t_k + 0) - x(t_k - 0). \end{cases} \tag{1}$$

where $(t, x) \in \Omega$.

Equations like (1) can be solved by the usual techniques of differential equations since these are piece-wise smooth equations.

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***Reformulation of the definition of impulsive systems***

We define an ascending step function $\tau : R \to Z$ with unit jumps at the impulse points:

$$\tau(t) := \quad k-1 \quad \text{if } t_{k-1} \le t < t_k,\ \forall\, k \in S_Z, \quad (2)$$

**Assumtion 1.** *Bainov assumes that the set of impulse points $S_H \subset R$ does not have any accumulation point in $R$. Hence $[-M, M] \cap S_H \subset R,\ \forall\, M \in R^+$ is a finite set.*

**Theorem 1.** *The ascending singular pure jumping function $\tau : R \to Z \subset R$ defined by equation (2) is well defined under Bainov's assumption 1 and $-\infty < \tau(t) < \infty,\ \forall\, t \in R$. The set of discontinuity points of $\tau$ is $S_H \subset R$ and each jump is $1 = \tau(t_k + 0) - \tau(t_k - 0),\ \forall\, t_k \in S_H$.*

**Corollary 1.** *The function $\tau : R \to R$ is a singular ascending function in t which means as an ascending function it is differentiable almost everywhere and*

$$\frac{d\tau(t)}{dt} = 0 \quad \textit{almost everywhere.}$$

*The singular ascending function $\tau$ defines a singular measure $\tau$ on the Borel sets of $R$. The domain of function $g$ is extended to the $\tilde{g} : \Omega \to R^n$ from the set $S_H \times \Omega_0$.*

With measure $\tau$ and $\tilde{g}$, equation (1) can be rewritten in integral form:

$$x(t) = x_0 + \int_{t_0}^{t} \left( f(s, x(s))ds + \tilde{g}(s, x(s))d\tau \right),$$
$$t \in T.\ x(t_0) = x_0. \qquad (3)$$

**Conclusion 1.** *Equation (3) is an equivalent formulation of equation (1) hence all impulsive systems have a formulation of (3). However, if we redefine Bainov's impulsive systems with impulses described by ascending singular functions, we have cases which are not in Bainov and his group's category. All Bainov type impulsive systems are formulated with non-continuous singular ascending functions (pure jumping functions) but it is known that there are continuous singular ascending functions (Nagy, 1965).*

***Impulsive Delayed Differential Equations***

The model developed by Bainov and his group (Ballinger and Liu, 1999) for delayed impulsive systems uses the same impulse points for the impact of an impulse and for the occurrence of impulses. Consider the case where a stone breaks the break fluid container at an impulse time point; there is no assurance that the break will be lost at another impulse time point. Hence, the time of the impact of an impulse on the system dynamics should be independent from the set $S_H$ of impulse time points. Let us see some simple examples to demonstrate what this would require.

Let the right side of the impulsive differential equation be defined as follows: Let $[a, b] \supset (\alpha, \beta),\ \eta := \dfrac{\alpha + \beta}{2}$ and let $\alpha - h =: \gamma \in (a, b],\ h > 0$. Then let

$$\begin{aligned} &f(x(t - \vartheta(t)), x(t)) := x(t) + x(t - \vartheta(t)),\\ &g(x(t - \vartheta(t)), x(t)) := 2x(t) + x(t - \vartheta(t)),\\ &\forall\, t \in (\alpha, \beta),\ S_H \cap (\alpha, \beta) = \emptyset,\ \&\ \gamma \in S_H. \end{aligned} \qquad (4)$$

Let the right continuous solution of the initial value problem of equation (4) be

$$\begin{aligned} x(t) = &x_0 + \int_{t_0}^{t} f(x(s - \vartheta(s)), x(s))ds +\\ &\int_{t_0}^{t} g(x(s - \vartheta(s)), x(s))d\tau, \quad t \in [t_0, b].\\ &x(t_0) = x_0,\ t_0 \in (a, \gamma) \setminus S_H. \end{aligned} \qquad (5)$$

Assume that $x(\gamma - 0) = 1,\ x(\gamma + 0) = -1$.
Let $\vartheta$ be continuous ascending function, $\vartheta(t) < t,\ \forall\, t \in (\alpha, \beta)$. Let $u(t) := t - \vartheta(t) \Rightarrow \vartheta(t) = t - u(t)$.

We will now show some simple examples to demonstrate that delay equations may lead to differential equations with measurable right sides.

1. Let $u(t) := \gamma - \varepsilon(t - \alpha)(\eta - t)(\beta - t)\ \forall\, t \in (\alpha, \beta)$. This will give $u(\alpha) = u(\eta) = u(\beta) = \gamma$ hence $x(\eta - 0) = 1 \neq -1 = x(\eta + 0)$. Therefore $f(x(t - \vartheta(t)), x(t))$ has both left and right limits which are not the same. Hence $f$ is measurable and not a continuous function of t in $[a, b]$ and $\vartheta$ is ascending with suitable selection of $\varepsilon$.

2. Let $t_j := \eta - \dfrac{\beta - \alpha}{2j},\ 1 \le j < \infty,\ t_j \nearrow \eta$ if $j \to \infty$. Then

$$u(t) := \begin{cases} \gamma + & \dfrac{(-1)^j}{2^2(\eta - \alpha)}(t - t_j)^2(t_{j+1} - t)^2,\\ & \text{if } \ t_j \le t \le t_{j+1},\ j \in \mathbb{N}.\\ 0 & \text{otherwise.} \end{cases}$$

The function $u(t) = \gamma,\ \forall\, t = t_i$ and $u(t) < \gamma$ if $t_i < t < t_{i+1}$ and $i$ is odd and $u(t) > \gamma$ if $t_i < t < t_{i+1}$ and $i$ is even. Hence $x(t_i - 0) = -1\ \&\ x(t_i + 0) = 1$ if $i$ is odd and $x(t_i - 0) = 1\ \&\ x(t_i + 0) = -1$ if $i$ is even.

Hence $\limsup_{t\nearrow\eta} x(t-\vartheta(t)) = 1$ and $\liminf_{t\nearrow\eta} x(t-\vartheta(t)) = -1$ .
The delayed solution $x(t-\vartheta(t))$ with this delay has no left limit hence no limit at $\eta \in (\alpha, \beta)$. What is more, there is no limit at $t_j$, $\forall\ 1 \le j < \infty$. Hence $f$ is measurable and not continuous function of t in $[a, b]$.

3. Continuous descending delay leads to bijective mapping of the impulse points hence in this case there are no accumulation points of the images of impact points but the statement about measurable right side remains valid.

**Conclusion 2.** *Based on the examples we will study the existence of the solutions of extended impulsive differential equations with the following assumptions:*

1. *The timing of the impulses will be defined by an ascending singular function of bounded variation $\tau$ in any bounded interval;*
2. *To support the delayed equations, the right side of the impulsive equation will be composed with $f(t, x)$ & $g(t, x) : \Omega \to R^n$, continuous or local Lipschitz functions in x for each fixed t, and measurable in t for each fixed x like Charatheodory's existence theorems are formulated.*
3. *The solutions will be functions of bounded variations.*

*The impulsive differential equations described in this conclusion will be referred to as extended impulsive differential equations.*

## EXPRESSION OF THE IMPULSIVE INTEGRAL EQUATIONS IN TERMS OF ONE MEASURE

The right side of integral equation (5) is a sum of integrals of two measures. In a differential equation (or the equivalent integral equation) an integral with one measure is expected. The singular ascending function $\tau$ is part of the impulsive model for being the impulse time controller. The details are presented in (Lipcsey et al., 2019b).

$\tau$ will play important role in our analysis therefore we will need some notations:

Let the domain of the time range of observation of the process be $[a, b_\lambda] \subset T$. Then $\tau : [a, b_\lambda] \to R^+$. Let $\nu_\lambda(t) := t + \tau(t)$ which is a strictly ascending function $\nu_\lambda : [a, b_\lambda] \to [a, b_\nu]$ with $b_\nu := b + \tau(b)$. As an ascending function $\nu_\lambda$ has a left- and a right-continuous versions:

$$\begin{aligned} \mu_{\lambda,-}(t) &:= \nu_\lambda(t-0),\ \forall\, t \in [a, b_\lambda] \\ \mu_{\lambda,+}(t) &:= \nu_\lambda(t+0),\ \forall\, t \in [a, b_\lambda] \end{aligned} \tag{6}$$

**Definition 1.** As ascending functions, $\tau$ and hence $\nu_\lambda$ have a countable set of discontinuity points denoted by $D_\lambda^\lambda := \{t \,|\, \tau(t-0) \neq \tau(t+0)\} = \{t \,|\, \nu_\lambda(t-0) \neq \nu_\lambda(t+0)\}$. Hence both $\tau$ & $\nu_\lambda$ are continuous in $[a, b_\lambda] \setminus D_\lambda^\lambda$.

The images of the discontinuity points in $D_\lambda^\lambda \subset [a, b_\lambda]$ are closed intervals $[\nu(t-0), \nu(t+0)] = [\mu_{\lambda,-}(t), \mu_{\lambda,+}(t)] \subset [a, b_\nu]$, $\forall\ t \in D_\lambda^\lambda$. The sets related to discontinuity of $\nu$ as function of $t \in [a, b_\lambda]$ are as follows:

**Definition 2.** Let $D_\lambda^\nu := \{[\mu_{\lambda,-}(t), \mu_{\lambda,+}(t)] \,|\, t \in D_\lambda^\lambda\} \subset \mathcal{P}([a, b_\nu])$ (where $\mathcal{P}([a, b_\nu])$ denotes the power set of $[a, b_\nu]$) and let $\mathcal{D}_\lambda^\nu := \bigcup_{t \in D_\lambda^\lambda} [\mu_{\lambda,-}(t), \mu_{\lambda,+}(t)] \subset [a, b_\nu]$.

The left and right continuous versions of $\nu_\lambda$ define a continuous mapping $\hat{\mu}_\lambda : [a, b_\nu] \to [a, b_\lambda]$ as follows:

**Definition 3.** Let

$$\begin{aligned} \hat{\mu}_\lambda(t) := s, \quad & t \in [\mu_{\lambda,-}(s), \mu_{\lambda,+}(s)], \\ & \forall\, t \in [a, b_\nu],\ \exists\, s \in [a, b_\lambda]. \end{aligned} \tag{7}$$

Since $\nu_\lambda$ is a non-continuous ascending function, the measure $\nu_\lambda$ is defined by the semi-ring of right open left closed intervals with continuity endpoints.

**Definition 4.** Let the semi-ring of left closed, right open intervals with continuity points as endpoints in $[a, b_\lambda] \setminus D_\lambda^\lambda$ with measure $\nu_\lambda$ be
$\mathcal{P}_{\nu,\,[a,b_\lambda],\,c} := \{[u, v) \mid u, v \in [a, b_\lambda] \setminus D_\lambda^\lambda\}$ and
$\nu_\lambda([s, t)) := (\mu_{\lambda,-}(t) - \mu_{\lambda,-}(s))$, $\forall\, [s, t) \in \mathcal{P}_{\nu,\,[a,b_\lambda],\,c}$.
Let the smallest $\sigma$-algebra containing the semi-ring be $\mathcal{B}_{\nu,\,[a,b_\lambda],\,c} := \sigma(\mathcal{P}_{\nu,\,[a,b_\lambda],\,c})$ with the extended measure $\nu_\lambda$ on it.

Since $\nu_\lambda(t) := t + \tau(t)$ and the measure $\nu_\lambda := \lambda + \tau$ both $\lambda$ & $\tau$ are absolute continuous with respect to $\nu_\lambda$ so:

$$\begin{aligned} \lambda(A) &= \int_A \frac{d\lambda}{d\nu_\lambda} d\nu_\lambda, \quad \forall\, A \in \mathcal{B}_{\nu,\,[a,b_\lambda],\,c}, \\ \tau(A) &= \int_A \frac{d\tau}{d\nu_\lambda} d\nu_\lambda, \quad \forall\, A \in \mathcal{B}_{\nu,\,[a,b_\lambda],\,c}. \end{aligned} \tag{8}$$

Using this in equation (3) we get an integral representation of the impulsive process of bounded variation in terms of a single measure:

$$\begin{aligned} & x(t) = x_0 + \\ & \int_{t_0}^{t} \left( f(s, x(s)) \frac{d\lambda}{d\nu_\lambda} + \tilde{g}(s, x(s)) \frac{d\tau}{d\nu_\lambda} \right) d\nu_\lambda, \\ & t \in T \ \&\ x(t_0) = x_0. \end{aligned} \tag{9}$$

***Properties of the Radon-Nikodym derivatives***

The integral in equation (3) composed with two measures is represented in equation (9) as an integral with one measure. The Radon-Nikodym derivatives of the two measures $\lambda$ & $\tau$ are used as written in equation (8). Both measures $\lambda$ & $\tau$ are absolute continuous with respect to $\nu_\lambda$ therefore both can be written as an integral of the Radon-Nikodym derivatives (Bass, 2011).
With the notations $\rho_\lambda = \dfrac{d\lambda}{d\nu_\lambda}$ & $\rho_\tau = \dfrac{d\tau}{d\nu_\lambda}$ the following important properties are formulated:

**Lemma 1.** *Let* $N_\tau^\lambda := \left\{ \dfrac{d\tau}{d\nu_\lambda} > 0 \right\}$ & $N_\lambda^\lambda := (N_\tau^\lambda)'$, *then*

1. $\lambda(N_\tau^\lambda) = 0 = \tau(N_\lambda^\lambda)$;
2. $\rho_\lambda + \rho_\tau = \dfrac{d(\lambda+\tau)}{d\nu_\lambda} = \dfrac{d\tau}{d\nu_\lambda} + \dfrac{d\lambda}{d\nu_\lambda} = 1$;
3. $\rho_\lambda(x) = 1, \ \forall x \in N_\lambda^\lambda, \ \rho_\tau(x) = 1, \ \forall x \in N_\tau^\lambda$;
4. $N_\lambda^\nu := \hat{\mu}_\lambda^{-1}(N_\lambda^\lambda)$
   $N_\tau^\nu := \hat{\mu}_\lambda^{-1}(N_\tau^\lambda)$
   *moreover* $N_\lambda^\nu \cup N_\tau^\nu = [a, b_\nu]$ *and* $N_\lambda^\nu \cap N_\tau^\nu = \emptyset$

For proof see lemma 2.1 and its proof in ((Lipcsey et al., 2019b). Since these constructions are discussed in details in the preceding paper (Lipcsey et al., 2019b), we will just summarize the concepts needed starting from the $\nu$-scale representation.

***From the total variation scale to bounded variation functions***

Assume that an interval $[a, b_\nu]$ of the total variation scale is given with a partition of Borel measurable sets $[a, b_\nu] = N_\lambda^\nu \cup N_\tau^\nu$ & $N_\lambda^\nu \cap N_\tau^\nu = \emptyset$. Let $\nu$ denote the Lebesgue measure on $[a, b_\nu]$. Then the basic concepts developed from functions of bounded variations are derived as follows:

**Definition 5.** The definition of basic mappings for time representation:

1. Let the time scale interval be $[a, b_\lambda]$ with $b_\lambda := a + \displaystyle\int_a^{b_\nu} \chi_{N_\lambda^\nu} d\nu$;
2. $\hat{\mu}_\lambda(s) := a + \displaystyle\int_a^s \chi_{N_\lambda^\nu} d\nu \in [a, b_\lambda], \ \forall \ s \in [a, b_\nu]$;
3. $\mu_{\lambda,-}(s) := \inf \hat{\mu}_\lambda^{-1}(\{s\}) \in [a, b_\nu], \ \forall \ s \in [a, b_\lambda]$;
4. $\mu_\lambda(s) := \mu_{\lambda,+}(s) := \sup \hat{\mu}_\lambda^{-1}(\{s\}) \in [a, b_\nu], \ \forall \ s \in [a, b_\lambda]$;
5. $D_\lambda^\lambda := \{t \,|\, \mu_{\lambda,-}(t) < \mu_{\lambda,+}(t)\}$ and let $D_\lambda^\nu := \{[\mu_{\lambda,-}(t), \mu_{\lambda,+}(t)] \,|\, t \in D_\lambda^\lambda\}$;
6. Let $\mathcal{D}_\lambda^\nu := \bigcup_{A \in D_\lambda^\nu} A$. Then $N_\lambda^\nu \subset [a, b_\nu] \setminus \mathcal{D}_\lambda^\nu$.
7. $N_\lambda^\lambda := \mu_{\lambda,-}^{-1}(N_\lambda^\nu)$ & $N_\tau^\lambda := \mu_{\lambda,-}^{-1}(N_\tau^\nu)$.

**Definition 6.** The definition of the $\tau$-based mappings for $\tau$-representation

1. Let the $\tau$-scale interval be $[0, b_\tau]$ with $b_\tau := \displaystyle\int_a^{b_\nu} \chi_{N_\tau^\nu} d\nu$;
2. $\hat{\mu}_\tau(s) := \displaystyle\int_a^s \chi_{N_\tau^\nu} d\nu \in [0, b_\tau], \ \forall \ s \in [a, b_\nu]$;
3. $\mu_{\tau,-}(s) := \inf \hat{\mu}_\tau^{-1}(\{s\}) \in [a, b_\nu]; \ \forall \ s \in [0, b_\tau]$;
4. $\mu_\tau(s) := \mu_{\tau,+}(s) := \sup \hat{\mu}_\tau^{-1}(\{s\}) \in [a, b_\nu], \ \forall \ s \in [0, b_\tau]$;
5. $D_\tau := \{t \,|\, \mu_{\tau,-}(t) < \mu_{\tau,+}(t)\}$ and let $D_\tau^\nu := \{[\mu_{\tau,-}(t), \mu_{\tau,+}(t)] \,|\, t \in D_\tau\}$;
6. Let $\mathcal{D}_\tau^\nu := \bigcup_{A \in D_\tau^\nu} A$. Then $N_\tau^\nu \subset [a, b_\nu] \setminus \mathcal{D}_\tau^\nu$.
7. $N_\lambda^\tau := \mu_{\tau,-}^{-1}(N_\lambda^\nu)$ & $N_\tau^\tau := \mu_{\tau,-}^{-1}(N_\tau^\nu)$.

**Theorem 2.** *1. The mappings defined in definition 5 fulfil:*

*(a)* $\hat{\mu}_\lambda : [a, b_\nu] \to [a, b_\lambda]$ *is an absolute continuous ascending function;*

*(b)* $\mu_{\lambda,-} : [a, b_\lambda] \to [a, b_\nu]$ *is a strictly ascending left continuous function while* $\mu_{\lambda,+} : [a, b_\lambda] \to [a, b_\nu]$ *is a strictly ascending right continuous function. Moreover, both mappings* $\mu_{\lambda,-}, \ \mu_{\lambda,+} : [a, b_\lambda] \setminus D_\lambda^\lambda \to [a, b_\nu] \setminus \mathcal{D}_\lambda^\nu$ *are continuous.*

*(c) Since* $\mu_{\lambda,-}$ & $\mu_{\lambda,+}$ *are ascending functions the set of their discontinuity points* $D_\lambda^\lambda$ *is a countable set.*

*(d) The mapping* $\hat{\mu}_\lambda : [a, b_\nu] \setminus \mathcal{D}_\lambda^\nu \to [a, b_\lambda] \setminus \mathcal{D}_\lambda$ *and* $\mu_{\lambda,-}, \ \mu_{\lambda,+} : [a, b_\lambda] \setminus \mathcal{D}_\lambda \to [a, b_\nu] \setminus \mathcal{D}_\lambda^\nu$ *fulfil the identities:*

$$(a.)\ \hat{\mu}_\lambda \circ \mu_{\lambda,-} = \hat{\mu}_\lambda \circ \mu_{\lambda,+} = id_{[a,b_\lambda]\setminus D_\lambda^\lambda}$$
$$\mu_{\lambda,-} \circ \hat{\mu}_\lambda = \mu_{\lambda,+} \circ \hat{\mu}_\lambda = id_{[a,b_\nu]\setminus \mathcal{D}_\lambda^\nu}$$
$$(b.)\ \hat{\mu}_\lambda \circ \mu_{\lambda,-} = \hat{\mu}_\lambda \circ \mu_{\lambda,+} = id_{[a,b_\lambda]}$$
$$\textit{if } \hat{\mu}_\lambda : [a, b_\nu] \to [a, b_\lambda] \ \&$$
$$\mu_{\lambda,-}, \ \mu_{\lambda,+} : [a, b_\lambda] \to [a, b_\nu].$$

*2. The mappings defined in definition 6 fulfil:*

*(a)* $\hat{\mu}_\tau : [a, b_\nu] \to [0, b_\tau]$ *is an absolute continuous ascending function;*

*(b)* $\mu_{\tau,-} : [0, b_\tau] \to [a, b_\nu]$ *is a strictly ascending left continuous function while* $\mu_{\tau,+} : [0, b_\tau] \to [a, b_\nu]$ *is a strictly ascending right continuous function. Moreover, both mappings* $\mu_{\tau,-}, \ \mu_{\tau,+} : [0, b_\tau] \setminus D_\tau \to [a, b_\nu] \setminus \mathcal{D}_\tau^\nu$ *are continuous.*

(c) *Since $\mu_{\tau,-}$ & $\mu_{\tau,+}$ are ascending functions the set of their discontinuity points $D_\tau$ is a countable set.*

(d) *The mapping $\hat{\mu}_\tau : [a, b_\nu] \setminus \mathcal{D}^\nu_\tau \to [0, b_\tau] \setminus D_\tau$ and $\mu_{\tau,-},\ \mu_{\tau,+} : [0, b_\tau] \setminus D_\tau \to [a, b_\nu] \setminus \mathcal{D}^\nu_\tau$ fulfil the identities:*

$$
\begin{aligned}
&(a.)\ \hat{\mu}_\tau \circ \mu_{\tau,-} = \hat{\mu}_\tau \circ \mu_{\tau,+} = id_{[0,b_\tau]\setminus D_\tau}\\
&\mu_{\tau,-} \circ \hat{\mu}_\tau = \mu_{\tau,+} \circ \hat{\mu}_\tau = id_{[a,b_\nu]\setminus \mathcal{D}^\nu_\tau}\\
&(b.)\ \hat{\mu}_\tau \circ \mu_{\tau,-} = \hat{\mu}_\tau \circ \mu_{\tau,+} = id_{[0,b_\tau]}\\
&\text{if } \hat{\mu}_\tau : [a, b_\nu] \to [0, b_\tau]\ \&\\
&\mu_{\tau,-},\ \mu_{\tau,+} : [0, b_\tau] \to [a, b_\nu].
\end{aligned}
$$

**Summary 1.** *Let us summarize the presentations and the mappings between them.*

**Representations:** *I. The Bainovian view of the impulsive differential equations is a description of the impulses which are represented in time scale on the interval $[a, b_\lambda]$;*
*II. The impulsive differential equations presenting the impulse process in interaction with the original system is the representation in $\tau$-scale on the interval $[a, b_\tau]$;*
*III. The absolute continuous representation of the process is the representation of $\nu$-scale on the interval $[a, b_\nu]$.*

**Mappings:** *There are six mappings:*

$$
\left.\begin{aligned}\mu_{\lambda,-} :\\ \mu_{\lambda,+} :\end{aligned}\right\} \quad [a, b_\lambda] \to [a, b_\nu]
\qquad
\left.\begin{aligned}\mu_{\tau,-} :\\ \mu_{\tau,+} :\end{aligned}\right\} \quad [a, b_\tau] \to [a, b_\nu] \tag{10}
$$

$$
\begin{aligned}
\hat{\mu}_\lambda &: [a, b_\nu] \to [a, b_\lambda];\\
\hat{\mu}_\tau &: [a, b_\nu] \to [a, b_\tau];
\end{aligned} \tag{11}
$$

*The functions $\mu_{\lambda,-},\ \mu_{\lambda,+}$ & $\mu_{\tau,-}$ & $\mu_{\tau,+}$ are left and right continuous versions of $\nu$ as functions of $t$ & $\tau$ respectively. The functions $\hat{\mu}_\lambda$ & $\hat{\mu}_\tau$ are absolute continuous mappings from $\nu$-scale to $t$ & $\tau$ scales respectively.*

**Measures:** ***The functions define the measures:*** *In these structures there are measures generated in t-scale by $id_{[a,b_\lambda]}$ & $\tau : [a, b_\lambda] \to R^+$, in $\tau$-scale generated by $\lambda$ & $id_{[a,b_\tau]} : [a, b_\tau] \to R^+$ and in $\nu$-scale by $id_{[a,b_\nu]} : [a, b_\nu] \to R^+$. The mapping $\nu$ generates measures on $[a, b_\lambda]$, & $[a, b_\tau]$ also.*

***Domains of the measures***

$[\mathbf{a}, \mathbf{b}_\lambda]$:

| | |
|---|---|
| $\lambda$ | $\mathcal{B}([a, b_\lambda], \lambda)$ *Borel sets;* |
| $\tau$ | $\mathcal{B}_{\lambda,\ [a,b_\lambda],\ c} := \sigma(\mathcal{P}_{\nu,\ [a,b_\lambda],\ c})$, |
| $\nu_\lambda$ | $\mathcal{P}_{\nu,\ [a,b_\lambda],\ c} := \{[u, v) \mid u, v \in [a, b_\lambda] \setminus D^\lambda_\lambda\}$ |

$[\mathbf{a}, \mathbf{b}_\tau]$:

| | |
|---|---|
| $\tau_\tau$ | $\mathcal{B}([a, b_\tau], \tau)$ *Borel sets;* |
| $\lambda_\tau$ | $\mathcal{B}_{\tau,\ [a,b_\tau],\ c} := \sigma(\mathcal{P}_{\tau,\ [a,b_\tau],\ c})$, |
| $\nu_\tau$ | $\mathcal{P}_{\tau,\ [a,b_\tau],\ c} := \{[u, v) \mid u, v \in [a, b_\tau] \setminus D^\tau_\tau\}$ |

$[\mathbf{a}, \mathbf{b}_\nu]$:

| | |
|---|---|
| $\nu$ | $\mathcal{B}([a, b_\nu], \nu)$ *Borel sets;* |
| $\nu^\nu_\lambda$ | $\mathcal{B}^\nu_{\lambda,\ [a,b_\nu],\ c} := \sigma(\mathcal{P}^\nu_{\lambda,\ [a,b_\nu],\ c})$ <br> $\mathcal{P}^\nu_{\lambda,\ [a,b_\nu],\ c} := \{[u, v) \mid u, v \in [a, b_\nu] \setminus \mathcal{D}^\nu_\lambda\}$ |
| $\nu^\nu_\tau$ | $\mathcal{B}^\nu_{\tau,\ [a,b_\nu],\ c} := \sigma(\mathcal{P}^\nu_{\tau,\ [a,b_\nu],\ c})$ <br> $\mathcal{P}^\nu_{\tau,\ [a,b_\nu],\ c} := \{[u, v) \mid u, v \in [a, b_\nu] \setminus \mathcal{D}^\nu_\tau\}$ |

*All measures in the table generated with intervals having continuity endpoints have continuous/atomic decomposition. Precisely the measures $\tau, \nu_\lambda$ in $[a, b_\lambda]$; $\lambda_\tau, \nu_\tau$ in $[a, b_\tau]$ and $\nu^\nu_\lambda, \nu^\nu_\tau$ in $[a, b_\nu]$ have atomic components.*

## EXISTENCE OF THE SOLUTION OF EXTENDED IMPULSIVE DIFFERENTIAL EQUATIONS

The impulsive differential equations were extended to have the following form: The Bainov's model of impulsive differential equations as described in equation (1) will be used as rewritten in equation (3).
Let the process evolve in a period of time $T \subset R$ which is in an open interval $\Omega \subset T \times R^n$. Let $f, g : \Omega \to R^n$ be measurable functions in the time variable t, and continuous/Lipschitz-continuous functions in the spatial variable x.
Let $\tau : T \to R^+$ be a singular ascending function of the time parameter $t$ with bounded variation on every closed bounded interval. Alternatively $\tau$ maybe a singular function of bounded variation on every closed bounded interval/compact subsets of $T$. If $\tau$ is of bounded variation as described, then its total variation $\tau_{tv}$ will be used as the singular "impulse set". It is important to see that $\tau$ may have a countable infinite set of jump points, where the total lengths of these jumps however must be finite.

**Solution of the extended impulsive differential equation**

The main results of this paper is a formulation of the extension of Caratheodory's existence theorem for the extended impulsive differential equations with measurable right side. The basis of our discussion is the approach presented in (Coddington and Levinson, 1955).

**Caratheodory's theorem**

We present Caratheodory's existence theorem in $R^n$ as it is presented in (Coddington and Levinson, 1955) for one dimension.

Let $\mathcal{S} \subset \Omega \subset R \times R^n$ be an open set. Let $f : S \to R^n$ be a function not necessarily continuous.

**Problem (E):** Find an interval $I \subset [a, b] \subset R$ and an absolute continuous function $\varphi : I \subset (a, b) \to R^n$ such that

$$\begin{aligned} &(t,\ \varphi(t)) \in S, \\ &\varphi'(t) = f(t,\ \varphi(t)), \quad \text{almost all } t \in I. \end{aligned} \tag{12}$$

Then the function $\varphi : I \to R^n$ is a solution of equation (12) in extended sense.

Charatheodory's existence theorem (Coddington and Levinson, 1955) targets finding a solutions to problem (E) with an initial value $(t_0, \xi) \in \mathcal{S},\ \varphi(t_0) = \xi$, when the right side is a measurable function in its variable $t$ for each fixed $x$, $\forall\ (t, x) \in \mathcal{S}$ where $\emptyset \neq (\alpha, \beta) = T \subset R$ & $\Omega_0 \subset R^n$ are open sets and $\Omega = T \times \Omega_0$.

**Definition 7.** Let a point $(t_0, \xi) \in \Omega$ be selected and let $\mathcal{R}_{\delta,\varepsilon}(t_0, \xi) := (t_0 - \delta,\ t_0 + \delta) \times B_\varepsilon(\xi) \subset \Omega,\ 0 < \delta, \varepsilon$ be a cylinder.

**Definition 8.** Let $\Omega := T \times \Omega_0,\ \emptyset \neq T := (\alpha, \beta) \subset R,\ \emptyset \neq \Omega_0 \subset R^n$ is an open set. Let $f : \Omega \to R^n$ be a measurable function. Let a point $(t_0, \xi) \in \Omega$ be selected. Let a cylinder $\mathcal{R}_{\delta,\varepsilon}(t_0, \xi) \subset \Omega,\ 0 < \delta,\ \varepsilon$ exist at $(t_0, \xi) \in \Omega$ with a function $m : (t_0 - \delta, t_0 + \delta) \to R^+ \setminus \{0\}$ to $f$ on the cylinder $\mathcal{R}_{\delta,\varepsilon}(t_0, \xi)$ such that $\|f(t, x)\| \leq m(t)\ \forall\ (t, x) \in \mathcal{R}_{\delta,\varepsilon}(t_0, \xi)$. Then $m$ is a dominating integrable function (D. I. F.) to $f$ on the cylinder $\mathcal{R}_{\delta,\varepsilon}(t_0, \xi)$.

**Theorem 3.** *(Caratheodory): Let $f : \mathcal{S} \to R^n$ be measurable in $t$ for each fixed $x$, and let it be continuous in $x$ for each fixed $t$, $\forall\ (t, x) \in \mathcal{S}$. Let $(t_0, \xi) \in \mathcal{S}$ be a fixed point and let a cylinder $\mathcal{R}_{\delta,\varepsilon}(t_0, \xi) \subset \mathcal{S}$ exist with a dominating integrable function $m : (t_0 - \delta,\ t_0 + \delta) \to R^+ \setminus \{0\}$ to $f$ on the cylinder $\mathcal{R}_{\delta,\varepsilon}(t_0, \xi)$ (definition 8). Then there exists a solution $\varphi$ of problem (E) in extended sense in an interval $(t_0 - \beta,\ t_0 + \beta), 0 < \beta \leq \delta$, such that $(t, \varphi(t)) \in \mathcal{R}_{\delta,\varepsilon}(t_0, \xi)$, $\forall\ t \in (t_0 - \beta, t_0 + \beta)$ and $\varphi(t_0) = \xi$.*

Using Charatheodory's theorem we can prove the existence of solution of the equation on $\nu$-scale as formulated in equation (13). Precisely

**Corollary 2.** *Let $f : N^\nu_\lambda \times \Omega_0 \to R^n$ & $g : N^\nu_\tau \times \Omega_0 \to R^n$ hence let $h := f\chi_{N^\nu_\lambda} + g\chi_{N^\nu_\tau}$ be measurable in $\sigma$ for each fixed $x$, and let it be continuous in $x$ for each fixed $\sigma$, $\forall\ (\sigma, x) \in \Omega$. Let $(\sigma_0, \xi_0) \in \Omega$ be a fixed point and let a cylinder $\mathcal{R}_{\delta,\varepsilon}(\sigma_0, \xi_0) \subset \Omega$ exist with a D.I.F. $m : (\sigma_0 - \delta, \sigma_0 + \delta) \to R^+ \setminus \{0\}$ on $\mathcal{R}_{\delta,\varepsilon}(\sigma_0, \xi_0)$ to $h$ (definition 8). Then there exists an interval $(\sigma_0 - \beta, \sigma_0 + \beta)$, $0 < \beta \leq \delta$ for the equation*

$$\begin{aligned} &\varphi(\sigma) = \xi_0 + \\ &\int_{\sigma_0}^{\sigma} \left(f(v, \varphi(v))\chi_{N^\nu_\lambda} + g(v, \varphi(v))\chi_{N^\nu_\tau}\right) d\nu^\nu_\lambda = \\ &\xi_0 + \int_{\sigma_0}^{\sigma} h(v, \varphi(v)) d\nu^\nu_\lambda \end{aligned} \tag{13}$$

*such that equation (13) has a solution $\varphi$ in that interval and $\varphi$ fulfils the condition $\varphi(\sigma_0) = \xi_0$.*

**Theorem 4.** *1. Let $h$ be $\mathcal{B}^\nu_{\lambda,\ [a,b_\nu],\ c}$-measurable. Let $\eta := \mu_{\lambda,+}(t) \in [a, b_\nu] \setminus \mathcal{D}^\nu_\lambda$, $\forall\ t \in [a, b_\lambda]$ and let $\varphi(\eta) := x_\lambda(\hat{\mu}_\lambda(\eta))\ \forall\ \eta \in [a, b_\nu] \setminus \mathcal{D}^\nu_\lambda$. Then the three formulations below are equivalent.*

$$\begin{aligned} \varphi(\eta) = \xi_0 + &\int_{\mu_{\lambda,+}(t_0)}^{\eta} \left(f(\hat{\mu}_\lambda(v), \varphi(v))\chi_{N^\nu_\lambda} + \right. \\ &\left. g(\hat{\mu}_\lambda(v), \varphi(v))\chi_{N^\nu_\tau}\right) d\nu^\nu_\lambda = \xi_0 + \\ &\int_{\mu_{\lambda,+}(t_0)}^{\mu_{\lambda,+}(t)} \left(f(\hat{\mu}_\lambda(v), x_\lambda(\hat{\mu}_\lambda(v)))\chi_{N^\nu_\lambda} + \right. \\ &\left. g(\hat{\mu}_\lambda(v), x_\lambda(\hat{\mu}_\lambda(v)))\chi_{N^\nu_\tau}\right) d\nu^\nu_\lambda = \xi_0 + \\ &\int_{t_0}^{t} \left(f(s, x_\lambda(s))\chi_{N_\lambda} + \right. \\ &\left. g(s, x_\lambda(s))\chi_{N^\lambda_\tau}\right) d\nu_\lambda = x_\lambda(t), \\ &\forall\ t \in [a, b_\lambda], \forall\ t_0 \in [a, b_\lambda], \\ &v \in [\mu_{\lambda,+}(t_0), \mu_{\lambda,+}(t)]. \end{aligned} \tag{14}$$

*By the identities in theorem 2, 1.d, the transformations $\mu_{\lambda,-} = \mu_{\lambda,+} : [a, b_\lambda] \setminus D_\lambda \to [a, b_\nu] \setminus \mathcal{D}^\nu_\lambda$ and $\hat{\mu}_\lambda : [a, b_\nu] \setminus \mathcal{D}^\nu_\lambda \to [a, b_\lambda] \setminus D_\lambda$ are invertible hence the $f$ components of the integrals mutually determine each-other. The $g$ function however in $\nu$-scale is a non-invertible image of the t-scale version.*

*2. Let $h$ be $\mathcal{B}^\nu_{\tau,\ [a,b_\nu],\ c}$-measurable. Let $\eta := \mu_{\tau,+}(\vartheta) \in [a, b_\nu] \setminus \mathcal{D}^\nu_\tau$, $\forall\ \vartheta \in [a, b_\tau] \setminus \mathcal{D}_\tau$ and let $\varphi(\eta) := x(\hat{\mu}_\tau(\eta))\ \forall\ \eta \in [a, b_\nu] \setminus \mathcal{D}^\nu_\tau$. Then the three formu-*

*lations below are equivalent.*

$$\varphi(\eta) = \xi_0 + \int\limits_{\mu_{\tau,+}(\vartheta_0)}^{\eta} \big(f(\hat{\mu}_\tau(v), \varphi(v))\chi_{N_\lambda^\nu} + g(\hat{\mu}_\tau(v), \varphi(v))\chi_{N_\tau^\nu}\big)\, d\nu_\tau^\nu = \xi_0 + \int\limits_{\mu_{\tau,+}(\vartheta_0)}^{\mu_{\tau,+}(\vartheta)} \big(f(\hat{\mu}_\tau(v), x(\hat{\mu}_\tau(v)))\chi_{N_\lambda^\nu} + g(\hat{\mu}_\tau(v), x(\hat{\mu}_\tau(v)))\chi_{N_\tau^\nu}\big)\, d\nu_\tau^\nu = \xi_0 + \int\limits_{\vartheta_0}^{\vartheta} (f(s, x_\tau(s))\chi_{N_\lambda} + g(s, x_\tau(s))\chi_{N_\tau^\lambda}\big)\, d\nu_\tau = x_\tau(\vartheta)$$
$$\forall\, \vartheta \in [a, b_\tau] \setminus \mathcal{D}_\tau, \forall\, \vartheta_0 \in [a, b_\tau] \setminus D_\tau, \quad v \in [\mu_{\tau,+}(\vartheta_0), \mu_{\tau,+}(\vartheta)]. \tag{15}$$

*By the identities in theorem 2, 2.d, the transformations $\mu_{\tau,-} = \mu_{\tau,+} : [a, b_\tau] \setminus D_\tau \to [a, b_\nu] \setminus \mathcal{D}_\tau^\nu$ and $\hat{\mu}_\tau : [a, b_\nu] \setminus \mathcal{D}_\tau^\nu \to [a, b_\tau] \setminus D_\tau$ are invertible hence the $g$ components of the integrals mutually determine each-other. The $f$ function however in $\nu$-scale is a non-invertible image of the $\tau$-scale version. However, if any of these equations has a solution then the other transformed versions also have and the corresponding transformed solutions are their solutions.*

***Remark*** 1. Relations (14) and (15) are valid in theorem 4 if $h$ is $\mathcal{B}^\nu_{\lambda,\,[a,b_\nu],\,c}$-measurable or $h$ is $\mathcal{B}^\nu_{\tau,\,[a,b_\nu],\,c}$-measurable respectively. When $h$ is integrable $\mathcal{B}([a, b_\nu], \nu)$-measurable then let $\nu_{h,\lambda}(A) := \int\limits_A h d\nu,\ \forall\, A \in \mathcal{B}^\nu_{\lambda,\,[a,b_\nu],\,c}$ and $\nu_{h,\tau}(A) := \int\limits_A h d\nu,\ \forall\, A \in \mathcal{B}^\nu_{\tau,\,[a,b_\nu],\,c}$ be signed measures on $\mathcal{B}^\nu_{\lambda,\,[a,b_\nu],\,c}$ and on $\mathcal{B}^\nu_{\tau,\,[a,b_\nu],\,c}$ absolute continuous with respect to $\nu^\nu_\lambda$ and absolute continuous with respect to $\nu^\nu_\tau$ respectively.

By the absolute continuity stated, $\mathcal{B}^\nu_{\lambda,\,[a,b_\nu],\,c}$- or $\mathcal{B}^\nu_{\tau,\,[a,b_\nu],\,c}$-measurable Radon Nikodym derivatives exists:

$$\tilde{h}_{\nu,\lambda} := \frac{d\nu_{h,\lambda}}{d\nu^\nu_\lambda} \Leftrightarrow \int\limits_A h d\nu = \int\limits_A \tilde{h}_{\nu,\lambda} d\nu^\nu_\lambda, \quad \forall\, A \in \mathcal{B}^\nu_{\lambda,\,[a,b_\nu],\,c}$$
$$\tilde{h}_{\nu,\tau} := \frac{d\nu_{h,\tau}}{d\nu^\nu_\tau} \Leftrightarrow \int\limits_A h d\nu = \int\limits_A \tilde{h}_{\nu,\tau} d\nu^\nu_\tau, \quad \forall\, A \in \mathcal{B}^\nu_{\tau,\,[a,b_\nu],\,c}. \tag{16}$$

Theorem 2.5 point (2) (a), (b) in (Lipcsey et al., 2019b) state that the Radon Nikodym derivative converts $\mathcal{B}([a, b_\nu], \nu)$-measurable integrable function to $\mathcal{B}^\nu_{\lambda,\,[a,b_\nu],\,c}$- or $\mathcal{B}^\nu_{\tau,\,[a,b_\nu],\,c}$-measurable integrands for which formulae (14) and (15) are valid in theorem 4 respectively. However, the integrands in the absolute continuous parts of the solutions will be composed from the Radon Nikodym derivatives of $f$ and $g$ instead of the functions $f$ and $g$. Theorem 2.1 in (Lipcsey et al., 2019a) states that the Radon Nikodym derivatives of $f$ and $g$ coincide with $f$ and $g$ on $[a, b_\lambda] \setminus D_\lambda$ or $[a, b_\tau] \setminus D_\tau$ respectively. Hence relations (14) and (15) are valid on the sets $[a, b_\lambda] \setminus D_\lambda$ or $[a, b_\tau] \setminus D_\tau$ respectively. The $\nu_\lambda$- and $\nu_\tau$-integrals on $D_\lambda$ & $D_\tau$ give the Bainovian impulses.

***The initial value problem***

We specify first the components of our extended impulsive differential equation which we wish to solve.

**Condition 1.** *We assume that $[a, b_\lambda]$, $[a, b_\tau]$ & $[a, b_\nu] \subset T$ are as used in this paper. Let $f : N^\lambda_\lambda \times \Omega_0 \subset [a, b_\lambda] \times \Omega_0 \to R^n$, $g : N^\tau_\tau \times \Omega_0 \subset [a, b_\tau] \times \Omega_0 \to R^n$, & $h : [a, b_\nu] \times \Omega_0 \to R^n$ are measurable functions and*

$$h(\sigma, \eta) := (f \circ \mu_{\lambda,-} \times \chi_{N^\nu_\lambda})(\sigma, \eta) + (g \circ \mu_{\tau,-} \times \chi_{N^\nu_\tau})(\sigma, \eta), \quad \forall\, (\sigma, \eta) \in [a, b_\nu] \times \Omega_0. \tag{17}$$

*Let the conditions specified in theorem 3 and corollary 2 be fulfilled by $f$ & $g$ equivalently let the solutions exist.*

The initial value problem specifies a time point either $t_0 \in (a, b_\lambda)$ or a time point $\vartheta_0 \in (a, b_\tau)$ and an initial value $\xi_0 \in \Omega_0$. These time specifications define $\sigma_0 \in (a, b_\nu)$ by either $\sigma_0 := \mu_{\lambda,-}(t_0)$ or $\sigma_0 := \mu_{\tau,-}(\vartheta_0)$ such that the solution of equation (13) fulfils the initial condition $\varphi(\sigma_0) = \xi_0$ with $(\sigma_0, \xi_0) \in \Omega$. Either $\sigma_0 \in N^\nu_\lambda$ or $\sigma_0 \in N^\nu_\tau$. Both mappings $\sigma_0 \in N^\nu_\lambda \to \hat{\mu}_\lambda(\sigma_0) = t_0 \in N^\lambda_\lambda \subset [a, b_\lambda]$ or $\sigma_0 \in N^\nu_\tau \to \hat{\mu}_\tau(\sigma_0) = \vartheta_0 \in N^\tau_\tau \subset [a, b_\tau]$ are bijective by theorem 2, point 1. (d)(b.) or point 2. (d)(b.) respectively. As a conclusion, either the t-scale solution $x_\lambda := \varphi \circ \mu_{\lambda,+}$ fulfils $x_\lambda(t_0) = \xi_0$ or the $\tau$-scale solution $x_\tau := \varphi \circ \mu_{\tau,+}$ fulfils $x_\tau(\vartheta_0) = \xi_0$ subject to the fulfilment of $\sigma_0 \in N^\nu_\lambda$ or $\sigma_0 \in N^\nu_\tau$ respectively. **Continuation of the solutions**

In corollary 2 we proved that if the condition of existence of local integrable dominator and conditions of continuity are fulfilled $\forall\, (\sigma_0, \xi_0) \in \Omega$ then there exists a $\delta_{(\sigma_0,\xi_0)} > 0$ such that the initial value problem with the initial condition $\varphi(\sigma_0) = \xi_0$ has a solution on the interval $[\sigma, \sigma + \delta_{(\sigma,x)})\ \ \forall(\sigma_0, \xi_0) \in \Omega$. This means that the solution cannot stop in $\Omega$ equivalently in $(a, b_\nu) \subset T$, it can stops at a time point $\sigma_{stop} \in [a, b_\nu] \subset T$ on a boundary point $(\sigma_{stop}, \varphi(\sigma_{stop})) \in \partial\Omega$ only. However, dealing with impulsive differential equations, we have to analyse the role of discontinuity points.

Using the notations in equation (1), the behaviour of impulsive systems is determined by $\mathcal{I}(t_k, y) := x(t_k - 0) + g(t_k, y)$, $x(t_k - 0) =: y \in \Omega_0$. If $\mathcal{I}(t_k, x(t_k - 0)) \notin \Omega_0$, the process stops at $(t_k, x(t_k - 0)) \in \Omega$. If $\mathcal{I}(t_k, y) \in \Omega_0$, $\forall (t_k, y) \in \Omega$ then all trajectory landing at a discontinuity point has continuation. If $\mathcal{I}$ is not onto $\Omega_0$ then any initial value problem with $(t_k, \xi_0) \in \{t_k\} \times (\Omega_0 \setminus \mathcal{R}(\mathcal{I}))$ (where $\mathcal{R}(\mathcal{I})$ is the range of $\mathcal{I}$) has a solution in an interval $[t_k, t_k + \delta), \delta > 0$, but no continuation backwards (there is no history).

**Uniqueness of solutions**

If the functions $f$ & $g$ are local Lipschitz-continuous for each fixed time parameter in their space variable then an initial value problem has locally unique solution by Caratheodory (see (Coddington and Levinson, 1955)). This means that a solution of an initial value problem can not split into more than one trajectory.

On the other hand trajectories may be connected together by impulse effects. This will make the flow of solutions to form a tree structure instead of a flow with trajectories coming from discontinuity points without history as leaves. This tree is directed, with orientation from the leaves to the root which is called in-tree or anti-arborescence (see (Fournier, 2013)).

## CONCLUSION

We established an existence theorem for impulsive differential equations with measurable right side which facilitates the analysis of delayed impulsive differential equations. The structure of solutions forms a tree structure, with orientation from the leaves to the root which is an in-tree or anti-arborescence. This gives wide range of modelling facilities by enabling to model and study mixing new generations in addition to studying flows of solutions.

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